Matrix Ernst Potentials and Orthogonal Symmetry for Heterotic String in Three Dimensions

Alfredo Herrera–Aguilar

Laboratory of Computing Techniques and Automation
Joint Institute for Nuclear Research, Dubna, M.R. 141980 Russia
e–mail: alfa@cv.jinr.dubna.su

and

Oleg Kechkin

Department of Electromagnetic Processes and Nuclear Interactions
Nuclear Physics Institute, Moscow State University, Moscow 119899 Russia
e–mail: kechkin@monet.npi.msu.su

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Abstract

A new coset matrix for low–energy limit of heterotic string theory reduced to three dimensions is constructed. The pair of matrix Ernst potentials uniquely connected with the coset matrix is derived. The action of the symmetry group on the Ernst potentials is established.

1 Review of Previous Results

In one–loop approximation the heterotic string theory leads to the effective action which describes matter fields coupled to gravity [1]:

$$S^{(D)}[G^{(D)}_{MN}, B^{(D)}_{MN}, \phi^{(D)}, A^{(D)}_{M}] = \int d^{(D)}x \mid G^{(D)} \mid^{\frac{1}{2}} \; e^{-\phi^{(D)}}(R^{(D)} + \phi^{(D)}_{;M}\phi^{(D);M} - \frac{1}{12} H^{(D)}_{MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I}_{MN} F^{(D)IMN}), \quad (1)$$

where

$$F^{(D)I}_{MN} = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M,$$
$$H^{(D)}_{MNP} = \partial_M B^{(D)}_{NP} - \frac{1}{2} A^{(D)I}_M F^{(D)I}_{NP} + \text{cycl. perms. of M,N,P.}$$

Here $G^{(D)}_{MN}$ is the D-dimensional metric, $B^{(D)}_{MN}$ is the antisymmetric Kalb–Ramond field, $\phi^{(D)}$ is the dilaton and $A^{(D)I}_M$ denotes a set ($I = 1,\, 2,\, ..., n$) of Abelian vector fields. For the self–consistent heterotic string theory $D = 10$ and $n = 16$ [2], but in this work, following [1], we shall leave these parameters arbitrary.

The action (1) can be generalized for the case of Yang–Mills gauge fields; it can also include mass, Gauss–Bonnet terms, etc. But only the simplest variant (1) of the theory possesses remarkable analytical properties which are important for our consideration.

In [1]–[2] it was shown that after the Kaluza–Klein compactification of $d = D - 3$ dimensions on a torus, the resulting theory is

$$S^{(3)}[g_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu, M] = \int d^3x \mid g \mid^{\frac{1}{2}} \; [R + \phi_{;\mu}\phi^{;\mu} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - e^{-2\phi} F^T_{\mu\nu} M^{-1} F^{\mu\nu} - \frac{1}{8} Tr\left(J^M\right)^2]. \quad (2)$$

Here the symmetric matrix M has the following structure

$$M = \begin{pmatrix} G^{-1} & G^{-1}(B+C) & G^{-1}A \\ (-B+C)G^{-1} & (G-B+C)G^{-1}(G+B+C) & (G-B+C)G^{-1}A \\ A^T G^{-1} & A^T G^{-1}(G+B+C) & I_n + A^T G^{-1} A \end{pmatrix} \quad (3)$$

with block elements defined by

$$G = (G_{pq} \equiv G^{(D)}_{p+2,q+2}),$$
$$B = (B_{pq} \equiv B^{(D)}_{p+2,q+2}),$$
$$A = (A^I_p \equiv A^{(D)I}_{p+2}),$$

where $C = \frac{1}{2}AA^T$ and $p, q = 1, 2, ..., d$. Matrix M satisfies the $O(d, d+n)$ group relation

$$MLM = L, \quad (4)$$

where

$$L = \begin{pmatrix} O & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & -I_n \end{pmatrix}; \tag{5}$$

thus $M \in O(d, d+n)/O(d) \times O(d+n)$.

The remaining 3–fields are defined in the following way: for dilaton and metric fields one has

$$\phi = \phi^{(D)} - \frac{1}{2} ln\, det\, G,$$
$$g_{\mu\nu} = e^{-2\phi}\left(G^{(D)}_{\mu\nu} - G^{(D)}_{p+2,\mu} G^{(D)}_{q+2,\nu} G^{pq}\right).$$

Then, the set of Maxwell strengths $F^{(a)}_{\mu\nu}$ $(a = 1, 2, ..., 2d + n)$ is constructed on $A^{(a)}_\mu$, where

$$A^p_\mu = \frac{1}{2} G^{pq} G^{(D)}_{q+2,\mu}$$
$$A^{I+2d}_\mu = -\frac{1}{2} A^{(D)I}_\mu + A^I_q A^q_\mu,$$
$$A^{p+d}_\mu = \frac{1}{2} B^{(D)}_{p+2,\mu} - B_{pq} A^q_\mu + \frac{1}{2} A^I_p A^{I+2d}_\mu.$$

Finally, the 3–dimensional axion

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2 A^a_\mu L_{ab} F^b_{\nu\rho} + \text{cycl. perms. of } \mu,\, \nu,\, \rho$$

depends on the 3–dimensional Kalb–Ramond field

$$B_{\mu\nu} = B^{(D)}_{\mu\nu} - 4 B_{pq} A^p_\mu A^q_\nu - 2\left(A^p_\mu A^{p+d}_\nu - A^p_\nu A^{p+d}_\mu\right).$$

The dimensionally reduced system (2) admits two simplifications. Namely, in three dimensions, the Kalb–Ramond field $B_{\mu\nu}$ becomes a non–dynamical variable and can be omitted [2]. Moreover, the fields A^a_μ can be dualized on–shell as follows

$$e^{-2\phi} M L F_{\mu\nu} = \frac{1}{2} E_{\mu\nu\rho} \nabla^\rho \psi; \tag{6}$$

so, the final system is defined by the quantities M, ϕ and ψ. As it had been established by Sen in [2], it is possible to introduce the matrix

$$\mathcal{M}_S = \begin{pmatrix} M + e^{2\phi}\psi\psi^T & -e^{2\phi}\psi & ML\psi + \frac{1}{2}\psi(\psi^T L\psi) \\ -e^{2\phi}\psi^T & e^{2\phi} & -\frac{1}{2}e^{2\phi}\psi^T L\psi \\ \psi^T LM + \frac{1}{2}e^{2\phi}\psi^T(\psi^T L\psi) & -\frac{1}{2}e^{2\phi}\psi^T L\psi & e^{-2\phi} + \psi^T LML\psi + \frac{1}{4}e^{2\phi}(\psi^T L\psi)^2 \end{pmatrix}, \tag{7}$$

in terms of which the action of the system adopts the standard chiral form

$$S^{(3)}[g_{\mu\nu}, \mathcal{M}_S] = \int d^3x \mid g \mid^{\frac{1}{2}} \left[R - \tfrac{1}{8} Tr\left(J^{\mathcal{M}_S}\right)^2\right], \tag{8}$$

where $J^{\mathcal{M}_S} = \nabla \mathcal{M}_S \mathcal{M}_S^{-1}$. This matrix is symmetric $\mathcal{M}_S = \mathcal{M}_S^T$ and satisfies the $O(d+1, d+n+1)$–group relation

$$\mathcal{M}_S \mathcal{L}_S \mathcal{M}_S = \mathcal{L}_S \tag{9}$$

with

$$\mathcal{L}_S = \left(\begin{array}{ccc} L & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right), \tag{10}$$

so that, $\mathcal{M}_S$ belongs to the coset $O(d+1, d+n+1)/O(d+1) \times O(d+n+1)$.

It is easy to see that the coset $O(d+1, d+n+1)/O(d+1) \times O(d+n+1)$ can be obtained from the coset $O(d, d+n)/O(d) \times O(d+n)$ by the replacement $d \to d+1$. At the same time, $\mathcal{M}_S$ has a quite different structure in comparison with M. Making use of these facts, one can hope that there is another chiral matrix $\mathcal{M}$ possessing the same structure that M with block components $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{A}$ of $(d+1) \times (d+1)$, $(d+1) \times (d+1)$ and $(d+1) \times n$ dimensions, respectively.

In this paper we show that such a matrix can actually be constructed. We establish that its block components allow to define two matrices ("matrix Ernst potentials") which permit to represent the theory under consideration in the Einstein–Maxwell (EM) form. At the end of the paper we study how the $O(d+1, d+n+1)$ group of transformations acts on the matrix Ernst potentials and establish the relations between its subgroups on the base of the discrete strong–weak coupling duality transformations (SWCDT) found in [2].

2 Matrix Ernst Potentials

We start from the consideration of the kinetic term of the matrix M

$$S^{(3)}[M] = -\tfrac{1}{8} \int d^3x \mid g \mid^{\frac{1}{2}} \; Tr \left(J^M \right)^2. \tag{11}$$

The Euler–Lagrange equation corresponding to (11) is

$$\nabla J^M = 0. \tag{12}$$

In terms of the block components G, B and A it reads

$$\begin{aligned} &\nabla J^G - \left(J^B \right)^2 + \nabla A \nabla A^T G^{-1} = 0, \\ &\nabla J^B - J^G J^B = 0, \\ &\nabla \left(G^{-1} \nabla A \right) - G^{-1} J^B \nabla A = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} J^G &= \nabla G \, G^{-1}, \\ J^B &= \left[\nabla B + \frac{1}{2} \left(A \nabla A^T - \nabla A A^T \right) \right] G^{-1}. \end{aligned} \tag{14}$$

Eqs. (13) are the motion equations for the action

$$S^{(3)}[G,B,A] = -\int d^3x \mid g \mid^{\frac{1}{2}} \; Tr\left\{\frac{1}{4}\left[\left(J^G\right)^2 - \left(J^B\right)^2\right] + \frac{1}{2}\nabla A^T\, G^{-1}\nabla A\right\}, \tag{15}$$

which is equivalent to (11) and can be obtained by straightforward but tedious algabraical calculations. (The coefficient $\frac{1}{4}$ can easily be established by comparison of Eqs. (11) and (15) in the case when $B = A = 0$).

One can introduce the matrix variable

$$X = G + B - \frac{1}{2}AA^T, \tag{16}$$

which was entered for the first time by Maharana and Schwartz in the case when $A = 0$ [1]; it defines, together with A, the most compact constrainless representation of the system:

$$S^{(3)}[X,A] = -\int d^3x \mid g \mid^{\frac{1}{2}} \; Tr\left[\tfrac{1}{4}\left(\nabla X + A\nabla A^T\right)G^{-1}\left(\nabla X^T + \nabla A A^T\right)G^{-1} + \tfrac{1}{2}\nabla A^T\, G^{-1}\nabla A\right], \tag{17}$$

where $G = \frac{1}{2}\left(X + X^T - AA^T\right)$. The form of this action is very similar to the stationary Einstein–Maxwell one [3]–[4]. Thus, in string gravity the matrix X formally plays the role of the gravitational potential $\mathcal{E}$, whereas the matrix A corresponds to the electromagnetic potential Φ of EM theory [5]. At the same time, one can notice a direct correspondence between the transposition of X and A on the one hand, and the complex conjugation of $\mathcal{E}$ and Φ, on the other. This analogy will be useful to study the symmetry group of string gravity in the last chapter of the paper.

For the complete theory, i.e., for the theory with nontrivial fields ϕ and ψ, the chiral current J^M does not preserve and one has the equation

$$\nabla J^M + 4e^{-2\phi}FF^TM^{-1} = 0 \tag{18}$$

instead of (12). The additional ϕ– and ψ–equations of motion are:

$$\begin{aligned} &\nabla^2\phi + \frac{1}{2}e^{2\phi}\nabla\psi^T M^{-1}\nabla\psi = 0, \\ &\nabla_\mu\left(e^{-2\phi}M^{-1}F^{\mu\nu}\right) = 0. \end{aligned} \tag{19}$$

They can be derived from the action

$$S^{(3)}[M,\phi,\psi] = -\int d^3x \mid g \mid^{\frac{1}{2}} \; Tr\left[(\nabla\phi)^2 - \frac{1}{2}e^{2\phi}\nabla\psi^T M^{-1}\nabla\psi + \frac{1}{8}Tr\left(J^M\right)^2\right] \tag{20}$$

by the usual variational procedure.

Our main aim is to represent the action (20) in a form similar to (11). We suppose that it can be done by the $[2(d+1)+n]\times[2(d+1)+n]$ matrix $\mathcal{M}$ defined by the block components $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{A}$ in the same way that the $[2d+n]\times[2d+n]$ matrix M is defined by G, B and A:

$$\mathcal{M} = \begin{pmatrix} \mathcal{G}^{-1} & \mathcal{G}^{-1}(\mathcal{B}+\mathcal{C}) & \mathcal{G}^{-1}\mathcal{A} \\ (-\mathcal{B}+\mathcal{C})\mathcal{G}^{-1} & (\mathcal{G}-\mathcal{B}+\mathcal{C})\mathcal{G}^{-1}(\mathcal{G}+\mathcal{B}+\mathcal{C}) & (\mathcal{G}-\mathcal{B}+\mathcal{C})\mathcal{G}^{-1}\mathcal{A} \\ \mathcal{A}^T\mathcal{G}^{-1} & \mathcal{A}^T\mathcal{G}^{-1}(\mathcal{G}+\mathcal{B}+\mathcal{C}) & I_n + \mathcal{A}^T\mathcal{G}^{-1}\mathcal{A} \end{pmatrix}. \tag{21}$$

This matrix also is a symmetric one and satisfies the $O(d+1,d+n+1)$–group relation

$$\mathcal{M}\mathcal{L}\mathcal{M}=\mathcal{L}, \tag{22}$$

where

$$\mathcal{L}=\left(\begin{array}{ccc} 0 & I_{d+1} & 0 \\ I_{d+1} & 0 & 0 \\ 0 & 0 & -I_n \end{array}\right), \tag{23}$$

and belongs to the coset $O(d+1,d+n+1)/O(d+1)\times O(d+n+1)$.

This hypothesis means that the action (20) can be expressed in the form

$$S^{(3)}[\mathcal{M}]=-\frac{1}{8}\int d^3x\mid g\mid^{\frac{1}{2}}\ Tr\left(J^{\mathcal{M}}\right)^2 \tag{24}$$

with $J^{\mathcal{M}}=\nabla\mathcal{M}\mathcal{M}^{-1}$; in view of (21), one can rewrite it as

$$S^{(3)}[\mathcal{G},\mathcal{B},\mathcal{A}]=-\int d^3x\mid g\mid^{\frac{1}{2}}\ Tr\left\{\frac{1}{4}\left[\left(J^{\mathcal{G}}\right)^2-\left(J^{\mathcal{B}}\right)^2\right]+\frac{1}{2}\nabla\mathcal{A}^T\,\mathcal{G}^{-1}\nabla\mathcal{A}\right\}. \tag{25}$$

In order to establish the explicit form of the matrix $\mathcal{M}$ one can procede as follows. On the one hand, it is useful to represent the column ψ in the form

$$\mathcal{L}_S\psi=\left(\begin{array}{c} u \\ v \\ s \end{array}\right). \tag{26}$$

Then Eq. (25) transforms to

$$S^{(3)}[G,B,A,\phi,u,v,s]=-\int d^3x\mid g\mid^{\frac{1}{2}}\ \{(\nabla\phi)^2+Tr\left[\frac{1}{4}\left((J^{\mathcal{G}})^2-(J^{\mathcal{B}})^2\right)+\frac{1}{2}\nabla\mathcal{A}^T\,\mathcal{G}^{-1}\nabla\mathcal{A}\right]-\frac{1}{2}e^{2\phi}(\nabla u+$$

$$(B+C)\nabla v+A\nabla s)^TG^{-1}(\nabla u+(B+C)\nabla v+A\nabla s)+\nabla v^TG\nabla v+\left(\nabla s-A^T\nabla v\right)^T\left(\nabla s-A^T\nabla v\right)\}. \tag{27}$$

On the other hand, the parametrization [1]

$$\mathcal{G}=\left(\begin{array}{cc} -f+\tilde{v}^TG\tilde{v} & \tilde{v}^TG \\ G\tilde{v} & G \end{array}\right),\quad \mathcal{B}=\left(\begin{array}{cc} 0 & \tilde{w}^T \\ -\tilde{w} & B \end{array}\right),\quad \mathcal{A}=\left(\begin{array}{c} \tilde{s}^T \\ A \end{array}\right), \tag{28}$$

with $\tilde{w}=\tilde{u}+B\tilde{v}$, leads to the following expressions for the 1–st and 3–rd terms of Eq. (25)

$$S^{(3)}[\mathcal{G}]=-\tfrac{1}{4}\int d^3x\mid g\mid^{\frac{1}{2}}\ Tr\left(J^{\mathcal{G}}\right)^2=-\int d^3x\mid g\mid^{\frac{1}{2}}\ \left\{\tfrac{1}{4}\left[f^{-2}(\nabla f)^2+Tr\left(J^{\mathcal{G}}\right)^2\right]-\tfrac{1}{2}f^{-1}\nabla v^TG\nabla v\right\}, \tag{29}$$

[1] The parametrization of the matrices G and B is written using the analogy between the theory under consideration and the theories with symplectic symmetry [6]– [7].

$$S^{(3)}[\mathcal{A}] = -\frac{1}{2}\int d^3x \mid g \mid^{\frac{1}{2}}\ Tr\left(\nabla A^T \mathcal{G}^{-1}\nabla A\right) = -\frac{1}{2}\int d^3x \mid g \mid^{\frac{1}{2}}\ [Tr\left(\nabla A^T G^{-1}\nabla A\right) -$$
$$f^{-1}\left(\nabla\tilde{s} - \nabla A^T\tilde{v}\right)^T\left(\nabla\tilde{s} - \nabla A^T\tilde{v}\right)]. \quad (30)$$

One can see that Eq. (29) gives the 1–st, 2–nd and 6–th terms of Eq. (27) if

$$f = e^{-2\phi} \quad \text{and} \quad \tilde{v} = v. \quad (31)$$

On the other hand Eq. (30) is equivalent to the 4–th and 7–th items of Eq. (25) if

$$\tilde{s} = -s + A^T v. \quad (32)$$

The second term of Eq. (25)

$$S^{(3)}[\mathcal{B}] = \frac{1}{4}\int d^3x \mid g \mid^{\frac{1}{2}}\ Tr\left(J^{\mathcal{B}}\right)^2 = \int d^3x \mid g \mid^{\frac{1}{2}}\ \left\{\frac{1}{4}Tr\left(J^{\mathcal{B}}\right)^2 + \frac{1}{2}f^{-1}\times\right.$$
$$\left.\left[\nabla\left(\tilde{u} - \frac{1}{2}As\right) + (B+C)\nabla v + A\nabla s\right]^T G^{-1}\left[\nabla\left(\tilde{u} - \frac{1}{2}As\right) + (B+C)\nabla v + A\nabla s\right]\right\} \quad (33)$$

corresponds to the remaining 3–rd and 5–th items of Eq. (27) if

$$\tilde{u} = u + \frac{1}{2}As. \quad (34)$$

Thus, the block components of the matrix $\mathcal{M}$ are defined by Eqs. (28), (31), (32) and (34). Consequently, the matrices $\mathcal{G}$ and $\mathcal{B}$ are

$$\mathcal{G} = \left(\begin{array}{cc} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{array}\right), \qquad \mathcal{B} = \left(\begin{array}{cc} 0 & \tilde{w}^T \\ -w & B \end{array}\right), \quad (35)$$

where $w = u + Bv + \frac{1}{2}As$. Finally, for the matrix Ernst potentials $\mathcal{X}$ and A one has

$$\mathcal{X} = \left(\begin{array}{cc} -e^{-2\phi} + v^T X v - v^T A s - \frac{1}{2}s^T s & v^T X + u^T + s^T A^T \\ X v - u & X \end{array}\right),$$
$$\mathcal{A} = \left(\begin{array}{c} -s^T + v^T A \\ A \end{array}\right). \quad (36)$$

3 Matrix Ehlers–Harrison Transformations

In this section we establish the action of the symmetry group $O(d+1, d+n+1)$ on the matrix Ernst potentials $\mathcal{X}$ and $\mathcal{A}$. It is evident that the action

$$S^{(3)}[g_{\mu\nu}, \mathcal{X}, \mathcal{A}] = -\int d^3x \mid g \mid^{\frac{1}{2}}\ \left\{R - Tr\left[\frac{1}{4}\left(\nabla\mathcal{X} + \mathcal{A}\nabla\mathcal{A}^T\right)\mathcal{G}^{-1}\left(\nabla\mathcal{X}^T + \nabla\mathcal{A}\mathcal{A}^T\right)\mathcal{G}^{-1} + \frac{1}{2}\nabla\mathcal{A}^T\mathcal{G}^{-1}\nabla\mathcal{A}\right]\right\}, \quad (37)$$

where $\mathcal{G} = \frac{1}{2}\left(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T\right)$, is invariant under the "rotation"

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_0\mathcal{H},\\ \mathcal{X} &= \mathcal{X}_0,\end{aligned} \tag{38}$$

where $\mathcal{H}\mathcal{H}^T = I_n$; this map generalizes the duality rotation of the electromagnetic sector in the stationary EM theory [8]. One can also see that the "scaling"

$$\begin{aligned}\mathcal{A} &= \mathcal{S}^T\mathcal{A}_0,\\ \mathcal{X} &= \mathcal{S}^T\mathcal{X}_0\mathcal{S},\end{aligned} \tag{39}$$

where $det\mathcal{S} \neq 0$, corresponds to the scale transformation of EM system. The gauge transformation of the potential $\mathcal{A}$ reads

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_0,\\ \mathcal{X} &= \mathcal{X}_0 + \mathcal{R}_1\end{aligned} \tag{40}$$

with $\mathcal{R}_1^T = -\mathcal{R}_1$, whereas for the gauge shift of the potential $\mathcal{X}$ one obtains

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_0 + \mathcal{T}_1,\\ \mathcal{X} &= \mathcal{X}_0 - \mathcal{T}_1\mathcal{A}_0^T - \frac{1}{2}\mathcal{T}_1\mathcal{T}_1^T.\end{aligned} \tag{41}$$

These transformations are the matrix analogues of the shifts of the rotational and electromagnetic variables of the stationary EM theory.

In order to find nontrivial transformations one can use SWCDT [2]. This symmetry transformation

$$\mathcal{M} \rightarrow \mathcal{M}^{-1} \tag{42}$$

can be expressed in terms of the matrices $\mathcal{X}$ and $\mathcal{A}$ as follows

$$\begin{aligned}\mathcal{A} &\rightarrow -\left(\mathcal{X} + \mathcal{A}\mathcal{A}^T\right)^{-1}\mathcal{A},\\ \mathcal{X} &\rightarrow \left(\mathcal{X} + \mathcal{A}\mathcal{A}^T\right)^{-1}\mathcal{X}^T\left(\mathcal{X}^T + \mathcal{A}\mathcal{A}^T\right)^{-1}.\end{aligned} \tag{43}$$

Using this map it is possible to obtain new transformations from the known ones (38)–(41). However, the scaling matrix subgroups remain invariant ($\mathcal{H} \rightarrow H$ and $\mathcal{S} \rightarrow (\mathcal{S}^T)^{-1}$) under (43). It turns out that the shift subgroups give rise to the actually non–linear transformations

$$\begin{aligned}\mathcal{A} &= \left[1 + \left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right)\mathcal{R}_2\right]^{-1}\mathcal{A}_0,\\ \left(\mathcal{X} + \mathcal{A}\mathcal{A}^T\right)^{-1} &= \left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right)^{-1} + \mathcal{R}_2,\end{aligned} \tag{44}$$

where $\mathcal{R}_2^T = -\mathcal{R}_2$, and

$$\begin{aligned}\mathcal{A} &= \left[1 + \mathcal{A}_0\mathcal{T}_2^T + \frac{1}{2}\left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right)\mathcal{T}_2\mathcal{T}_2^T\right]^{-1}\left[\mathcal{A}_0 + \left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right)\mathcal{T}_2\right],\\ \mathcal{X} + \mathcal{A}\mathcal{A}^T &= \left[1 + \mathcal{A}_0\mathcal{T}_2^T + \frac{1}{2}\left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right)\mathcal{T}_2\mathcal{T}_2^T\right]^{-1}\left(\mathcal{X}_0 + \mathcal{A}_0\mathcal{A}_0^T\right).\end{aligned} \tag{45}$$

Formula (44) generalizes the Ehlers transformation [9] for the string system, whereas Eq. (45) provides the matrix analogue of the Harrison ("charging") transformation [8].

At the end of the paper we would like to remark that the relations (38)–(41) and (44)-(45) form the full set of transformations of the $O(d+1, d+n+1)$ group. Actually, the general $O(d+1, d+n+1)$ matrix $\mathcal{K}$, which defines the authomorphism $\mathcal{M} \rightarrow \mathcal{K}^T\mathcal{M}\mathcal{K}$, can be represented in the following form

$$\mathcal{K} = \mathcal{K}_{\mathcal{T}_2}\mathcal{K}_{\mathcal{R}_2}\mathcal{K}_{\mathcal{S}}\mathcal{K}_{\mathcal{H}}\mathcal{K}_{R_1}\mathcal{K}_{\mathcal{T}_1}, \tag{46}$$

where

$$\mathcal{K}_{\mathcal{T}_2} = \begin{pmatrix} I_{d+1} & 0 & 0 \\ K_{\mathcal{T}_2} & I_{d+1} & \mathcal{T}_2 \\ \mathcal{T}_2^T & 0 & I_n \end{pmatrix}, \quad \mathcal{K}_{\mathcal{R}_2} = \begin{pmatrix} I_{d+1} & 0 & 0 \\ \mathcal{R}_2 & I_{d+1} & 0 \\ 0 & 0 & I_n \end{pmatrix},$$

$$\mathcal{K}_{\mathcal{S}} = \begin{pmatrix} (\mathcal{S}^T)^{-1} & 0 & 0 \\ 0 & \mathcal{S} & 0 \\ 0 & 0 & I_n \end{pmatrix}, \quad \mathcal{K}_{\mathcal{H}} = \begin{pmatrix} I_{d+1} & 0 & 0 \\ 0 & I_{d+1} & \\ 0 & 0 & \mathcal{H} \end{pmatrix}, \tag{47}$$

$$\mathcal{K}_{\mathcal{R}_1} = \begin{pmatrix} I_{d+1} & \mathcal{R}_1 & 0 \\ 0 & I_{d+1} & 0 \\ 0 & 0 & I_n \end{pmatrix}, \quad \mathcal{K}_{\mathcal{T}_1} = \begin{pmatrix} I_{d+1} & K_{\mathcal{T}_1} & T_1 \\ 0 & I_{d+1} & 0 \\ 0 & \mathcal{T}_1^T & I_n \end{pmatrix}.$$

Here $K_{\mathcal{T}_2} = \frac{1}{2}\mathcal{T}_2\mathcal{T}_2^T$ and $K_{\mathcal{T}_1} = \frac{1}{2}\mathcal{T}_1\mathcal{T}_1^T$; moreover, $[\mathcal{K}_{\mathcal{T}_2}, \mathcal{K}_{\mathcal{R}_2}] = [\mathcal{K}_{\mathcal{R}_1}, \mathcal{K}_{\mathcal{T}_1}] = [\mathcal{K}_{\mathcal{S}}, \mathcal{K}_{\mathcal{H}}] = 0$, and under the map (42) one has

$$\begin{aligned} &\mathcal{K}_{\mathcal{T}_1} \rightarrow \mathcal{K}_{\mathcal{T}_2}, \\ &\mathcal{K}_{\mathcal{H}} \rightarrow \mathcal{K}_{\mathcal{H}}, \\ &\mathcal{K}_{\mathcal{S}} \rightarrow \mathcal{K}_{(\mathcal{S}^T)^{-1}}, \\ &\mathcal{K}_{\mathcal{R}_1} \rightarrow \mathcal{K}_{\mathcal{R}_2}, \end{aligned} \tag{48}$$

where $\mathcal{R}_1 \rightarrow \mathcal{R}_2$ and $\mathcal{T}_1 \rightarrow -\mathcal{T}_2$. Thus, the complete $O(d+1, d+n+1)$ group consists of six subgroups defined by the matrices $\mathcal{H}$, $\mathcal{S}$; $\mathcal{T}_1$, $\mathcal{T}_2$; $\mathcal{R}_1$, $\mathcal{R}_2$. This subgroups are the same ones that we have considered above (see Eqs. (38)–(41) and (44)–(45)).

4 Conclusion and Discussion

In this paper we study the $O(d+1, d+n+1)$–symmetric low–energy limit of heterotic string theory reduced to three dimensions. It is shown that such a theory can be represented in terms of the $(d+1) \times (d+1)$ matrix $\mathcal{X}$ and $(d+1) \times n$ matrix $\mathcal{A}$. These matrices appear to be the analogues of the gravitational and electromagnetic potentials ($\mathcal{E}$ and Φ, respectively) of the stationary EM theory. The matrices $\mathcal{G} = \frac{1}{2}\left(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T\right)$, $\mathcal{B} = \frac{1}{2}\left(\mathcal{X} - \mathcal{X}^T\right)$ and $\mathcal{A}$ define the chiral matrix $\mathcal{M} \in O(d+1, d+n+1)/O(d+1) \times O(d+n+1)$ of the theory in the same way that matrices G, B and A (constructed on the extra components of the metric, Kalb–Ramond and electromagnetic fields, respectively) define the coset matrix $M \in O(d, d+n)/O(d) \times O(d+n)$.

It is established that the $O(d+1, d+n+1)$ symmetry group can be decomposed into six subgroups using the strong–weak coupling duality transformation. It turns out that two subgroups (the rescaling of the potentials $\mathcal{X}$ and $\mathcal{A}$) are invariant under SWCDT. At the same time, the remaining transformations combine into two pairs which map one into another under SWCDT. We show that the gauge shift of $\mathcal{X}$ maps into the matrix Ehlers tranformation, whereas the shift of $\mathcal{A}$ maps into the matrix Harrison one.

All subgroups of transformations are written in quasi–Einstein–Maxwell form. This fact remarks the analogy between the string gravity system with orthogonal symmetry, on the one hand, and the EM theory, on the other, in the 3–dimensional case.

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